A quantitative version of the transversality theorem

Andrew Murdza and Khai T. Nguyen

Department of Mathematics, North Carolina State University
e-mails: apmurdza@ncsu.edu, khai@math.ncsu.edu

Dedicated to Professor Duong Minh Duc on the occasion of his 70th birthday

March 22, 2022

Abstract

The present paper studies a quantitative version of the transversality theorem. More precisely, given a continuous function $g \in \mathcal{C}([0,1]^d, \mathbb{R}^m)$ and a global smooth manifold $W \subset \mathbb{R}^m$ of dimension p, we establish a quantitative estimate on the $(d+p-m)$-dimensional Hausdorff measure of the set $\mathcal{Z}_W^g = \left\{x \in [0,1]^d : g(x) \in W\right\}$. The obtained result is applied to quantify the total number of shock curves in weak entropy solutions to scalar conservation laws with uniformly convex fluxes in one space dimension.

Keyword. Transversality lemma, quantitative estimates, conservation laws

AMS Mathematics Subject Classification. 46T20, 35L67

1 Introduction

Given two smooth manifolds X of dimension d and Y of dimension m, let $g : X \to Y$ be a $\mathcal{C}^1$ map. For any smooth submanifold W of Y, we say that function g is transverse to W and write $g \pitchfork W$ if

$$(dg)_p(T_pX) + T_{g(p)}(W) \;=\; T_{g(p)}(Y) \quad \text{for all } p \in g^{-1}(W).$$

The transversality lemma, which is the key to proving Thom's transversality theorem [6], shows that the set of transversal maps is dense [5]. In particular, given a smooth manifold $W \subset \mathbb{R}^m$ of dimension p, for any continuous function $g : [0,1]^d \mapsto \mathbb{R}^m$ and any $\varepsilon > 0$, there exists a $\mathcal{C}^1$ function $g_\varepsilon : [0,1]^d \to \mathbb{R}^m$ such that

$$\|g_\varepsilon - g\|_{\mathcal{C}^1} \;\leq\; \varepsilon \qquad \text{and} \qquad g_\varepsilon \pitchfork W.$$

For every $h \in \mathcal{C}\big([0,1]^d, \mathbb{R}^m\big)$, consider the set

$$\mathcal{Z}_W^h \;:=\; \Big\{x \in [0,1]^d : h(x) \in W\Big\}. \tag{1.1}$$

If h is smooth and transversal to W, then $\mathcal{Z}_W^h$ is a $(d+p-m)$-dimensional smooth manifold. Hence, its $(d+p-m)$-dimensional Hausdorff measure is finite. In this paper, we perform a quantitative analysis of the measure of $\mathcal{Z}^g_W$. Namely, how small can we make this measure, by an ε-perturbation of g? To formulate more precisely our result, given $g \in \mathcal{C}([0,1]^d, \mathbb{R}^m)$, define

$$\mathcal{N}^g_W(\varepsilon) \ := \inf_{\|h-g\|_{\mathcal{C}^0} \le \varepsilon} \mathcal{H}^{d+p-m}\left(\mathcal{Z}^h_W\right)$$

to be the smallest $(d+p-m)$-Hausdorff measure of $\mathcal{Z}^h_W$ among all functions $h \in \mathcal{C}\left([0,1]^d, \mathbb{R}^m\right)$ with $\|h-g\|_{\mathcal{C}^0} \le \varepsilon$. Relying on the concept of Kolmogorov ε-entropy [17], we will establish an upper bound on the number $\mathcal{N}^g_W(\varepsilon)$, for a general continuous function $g : [0,1]^d \to \mathbb{R}^m$. The result can be extended to the case of continuous functions $g : X \to Y$ where X, Y are global smooth manifolds and $W \subseteq Y$ is a smooth submanifold of Y. Specially, we obtain the following estimate for a class of Hölder continuous functions.

Theorem 1.1 *Assume that $p + d \ge m$ and $g \in \mathcal{C}^\alpha([0,1]^d, \mathbb{R}^m)$ is Hölder continuous with exponent $\alpha \in (0,1]$. Then for every $\varepsilon > 0$ sufficiently small, it holds*

$$\mathcal{N}^g_W(\varepsilon) \ \le \ C_W \cdot \left(\frac{\|g\|_{\mathcal{C}^{0,\alpha}}}{\varepsilon}\right)^{\frac{m-p}{\alpha}} \tag{1.2}$$

where the constant $C_W > 0$ depends only on W and $\|g\|_{\mathcal{C}^{0,\alpha}}$ is the Hölder norm of g.

In the scalar case ($d = m = 1$), the blow up rate $\left(\frac{1}{\varepsilon}\right)^{\frac{m-p}{\alpha}}$ with respect to ε is shown to be the best bound in terms of power function in Example 3.4. For the multi-dimensional cases ($d \ge 2$), this should be still true but the situation becomes considerably more technical. We leave this open.

In the second part of the paper, we give an application to conservation laws. For several classes of hyperbolic PDEs, one can prove that there exists an open dense set of initial data for which the solution develops at most a finite number of singularities [7, 11, 12, 13]. A natural question is to provide a quantitative estimate on this number. For example, consider the scalar conservation laws in one space dimension

$$u_t(t,x) + f\big(u(t,x)\big)_x \ = \ 0 \qquad\qquad (t,x) \in [0,\infty[\, \times \mathbb{R}, \tag{1.3}$$

with strictly convex flux f. In this case, there is a connection between (1.3) and the Hamilton-Jacobi equation which induces an explicit representation of solutions. Using this representation, Oleinik [18, 19, 20] shows that solutions of (1.3) are continuous, except on the union of an at most countable set of shock curves. Analogous results also established for solutions to genuinely nonlinear hyperbolic systems of conservation laws in [8, 14, 15, 16]. The structure and smoothness of solutions to (1.3) were studied in [11], using the concept of generalized characteristics. For a dense set of initial data, a stronger regularity property holds. Namely, the total number of shock curves is finite [13]. Here, in the spirit of metric entropy which was used in the study of the compactness estimates for solution sets [2, 3, 4, 9], we shall provide quantitative estimates on the number of shock curves in an entropy weak solution u to (1.3), which is a weak solution of (1.3) in the sense of distributions and satisfies an entropy criterion for admissibility

$$u(t,x-) \ \ge \ u(t,x+) \qquad \text{for } a.e.\ t > 0, x \in \mathbb{R}.$$

More precisely, assuming that $f \in \mathcal{C}^4(\mathbb{R})$ is uniformly convex, i.e.,

$$f''(u) \ \ge \ \lambda \ > \ 0 \qquad \text{for all } u \in \mathbb{R}. \tag{1.4}$$

For any given $\varepsilon > 0$ and $\bar{u} \in \mathbf{L}^1(\mathbb{R})$ with a compact support, we seek a perturbed initial datum $\bar{v} \in C_c^3(\mathbb{R})$, with $\|\bar{v} - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon$, such that the solution $v = v(t,x)$ of (1.3) with $v(0,\cdot) = \bar{v}$ has the total number of shocks bounded in terms of ε^{-1}. The next simplified theorem provides an upper bound on this number of shocks.

Theorem 1.2 *Let the flux function f be $\mathcal{C}^4$-smooth and satisfy (1.4). Given constants $R, V > 0$, assume that $\bar{u} \in \mathbf{L}^1(\mathbb{R})$ satisfies*

$$\mathrm{Supp}(\bar{u}) \subseteq [-R, R] \qquad \text{and} \qquad \mathit{Tot.Var.}\{\bar{u}\} \ < \ V. \tag{1.5}$$

Then, for some constant C, the following holds. For every $\varepsilon > 0$ sufficiently small, there exists $\bar{v} \in \mathcal{C}^3(\mathbb{R})$ with $\mathrm{Supp}(\bar{v}) \subseteq [-2R, 2R]$ and $\|\bar{v} - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon$, such that the entropy weak solution $v = v(t,x)$ of (1.3) with initial datum $v(0,\cdot) = \bar{v}$ satisfies

$$[\text{Total number of shock curves of } v] \ \leq \ \frac{C}{\lambda} \cdot \frac{R^4 V^5}{\varepsilon^4} + 4. \tag{1.6}$$

The proof of Theorem 1.2 relies on Theorem 1.1 and the observation that the total number of shock curves arising in the solution v is bounded by the total number of inflection points of the function $x \mapsto f'\big(\bar{v}(x)\big)$. Finally, we remark that the constant C is explicitly computed in (4.38) and the result is proved for $\mathcal{C}^3$-smooth f in Theorem 4.3.

The remainder of this paper is organized as follows. In Section 2, we recall basic concepts on the inverse of the minimal modulus of continuity and Komolgorov ε-entropy, and also include a necessary result on the partition of the unit cube into polytopes in $\mathbb{R}^d$. Section 3 contains a general result on an upper estimate for $\mathcal{N}_W^g(\varepsilon)$, while Section 4 provides a brief review on the scalar conservation laws with uniformly convex fluxes in one space dimension and extends Theorem 1.2 to the case of $\mathcal{C}^3$-smooth f.

2 Notations and preliminaries

Let $d \geq 1$ be an integer and D be a measurable subset of $\mathbb{R}^d$. Throughout the paper we shall denote by:

- $|\cdot|$ the Euclidean norm of $\mathbb{R}^d$;
- $B_d(a,r) = \{x \in \mathbb{R} : |x - a| < r\}$ the ball of radius r centered at $a \in \mathbb{R}^d$ and
$$B_d(U,r) \ = \ \bigcup_{a \in U} B_d(a,r) \qquad \text{for all } r \geq 0, U \subseteq \mathbb{R}^d;$$
- $\mathrm{Int}(D)$ the interior of D;
- $\mathrm{Diam}(D) = \sup_{x,y \in D} |x - y|$, the diameter of the set D in $\mathbb{R}^d$;
- $\chi_D = \begin{cases} 1 & \text{if} \quad x \in D \\ 0 & \text{if} \quad x \in \mathbb{R}^d \backslash D \end{cases}$ the characteristic function of a subset D in $\mathbb{R}^d$;

- $\mathcal{H}^s(A)$ the s-dimensional Hausdorff measure of A;
- $\#(S)$ the number of elements of any finite set S;
- $\mathbf{L}^1(\mathbb{R})$ the Lebesgue space of all (equivalence classes of) summable functions on $\mathbb{R}$, equipped with the usual norm $\|\cdot\|_{\mathbf{L}^1}$;
- $\mathbf{L}^\infty(\mathbb{R})$ the space of all essentially bounded functions on $\mathbb{R}$, equipped with the usual norm $\|\cdot\|_{\mathbf{L}^\infty}$;
- $\mathcal{C}^n(\mathbb{R})$, space of smooth functions on $\mathbb{R}$ having continuous derivatives $f', f'', \dots, f^{(n)}$, equipped with the usual norm $\|\cdot\|_{\mathcal{C}^n}$;
- Tot.Var.$\{g, I\}$ total variation of g over the open interval I in $\mathbb{R}$;
- Supp(u) the essential support of a function $u \in \mathbf{L}^\infty(\mathbb{R})$;
- $\lfloor x \rfloor := \max\{z \in \mathbb{Z} : z \leq x\}$ the integer part of x.

In order to obtain the estimate (1.2) for general continuous functions, let us introduce the inverse of the minimal modulus of a continuity.

Definition 2.1 *Given subsets $U \subseteq \mathbb{R}^d$ and $V \subseteq \mathbb{R}^m$, let $h : U \to V$ be continuous. The minimal modulus of continuity of h is given by*

$$\omega_h(\delta) \;=\; \sup_{x,y\in U, |x-y|\leq\delta} |h(y)-h(x)| \qquad \text{for all } \delta \in [0, \operatorname{diam}(U)]. \tag{2.7}$$

The inverse of the minimal modulus of continuity of h is the map $s \mapsto \Psi_h(s)$ is defined by

$$\Psi_h(s) \;:=\; \sup\{\delta \geq 0 : |h(x)-h(y)| \leq s \;\; \text{for all } |x-y| \leq \delta, x, y \in U\} \tag{2.8}$$

for all $s \geq 0$.

From the above definition, it is clear that $\Psi_h(s) = \infty$ for all $s \in [M_h, \infty[$ with $M_h := \sup_{x,y\in U} |h(x)-h(y)|$. In particular, if h is a constant function then $\Psi_h(s) = \infty$ for all $s \geq 0$. Otherwise, by the continuity of h, it holds

$$\Psi_h(0) \;=\; 0 \qquad \text{and} \qquad 0 \;<\; \Psi_h(s) \;\leq\; \operatorname{diam}(U) \qquad \text{for all } s \in]0, M_h[.$$

Moreover, $\Psi_h(\cdot) : [0,\infty[\to [0,\infty[$ is increasing and superadditive

$$\Psi_h(s_1+s_2) \;\geq\; \Psi_h(s_1) + \Psi_h(s_2) \qquad \text{for all } s_1, s_2 \geq 0.$$

If the map $\delta \mapsto \omega_h(\delta)$ is strictly increasing in $[0, \operatorname{diam}(U)[$ then Ψ_h is the inverse of ω_h, i.e.,

$$\Psi_h(s) \;=\; \omega_h^{-1}(s) \qquad \text{for all } s \in [0, M_h[.$$

In the case that h is Hölder continuous with an exponential $\alpha \in (0,1]$, for every $s > 0$ it holds

$$\Psi_h(s) \;\geq\; \left(\frac{s}{\|h\|_{\mathcal{C}^{0,\alpha}}}\right)^{\frac{1}{\alpha}} \qquad \text{with} \qquad \|h\|_{\mathcal{C}^{0,\alpha}} = \sup_{x,y\in U, x\neq y} \frac{|h(x)-h(y)|}{|x-y|^\alpha}. \tag{2.9}$$

Toward a sharp estimate on $\mathcal{N}_W^g(\varepsilon)$, we recall the concept of Kolmogorov ε-entropy [17] which has been studied extensively in a variety of literature and disciplines. It plays a central role in various areas of information theory and statistics, including nonparametric function estimation, density information, empirical processes and machine learning. It provides a tool for characterizing the rate of mixing of sets of small measure.

Definition 2.2 *Given a metric space (E, ρ), let K be a totally bounded subset of E. For any $\varepsilon > 0$, let $\mathbf{N}_\varepsilon(K|E)$ be the minimal number of sets in a covering of K by subsets of E having diameter no larger than 2ε. Then the ε-entropy of K is defined as*

$$\mathbf{H}_\varepsilon(K|E) \ := \ \log_2 \mathbf{N}_\varepsilon(K|E).$$

To complete this section, let us prove a simple lemma of the decomposition of a unit cube in $\mathbb{R}^d$ which will be used in the proof of Theorem 3.2.

Lemma 2.3 *Let $\square^d = [0,1]^d$ be a unit cube in $\mathbb{R}^d$. Then, $\square^d$ can be decomposed into $2^{d-1}d!$ polytopes Δ_k^d in $\mathbb{R}^d$ such that Δ_k^d has $(d+1)$ vertices for $k \in \{0, 1, \dots, 2^{d-1}d! - 1\}$.*

Proof. The decomposition of $\square^d$ can be done by using the induction process:

- If $d = 1$ then $\square^1$ is an interval $[0, 1]$.
- For $d \geq 2$, assume that $\square^{d-1}$ can be decomposed into $2^{d-2}(d-1)!$ polytopes Δ_ℓ^{d-1} in $\mathbb{R}^{d-1}$ such that Δ_ℓ^{d-1} has d vertices for $\ell \in \{0, 1, \dots, 2^{d-2}(d-1)! - 1\}$. Observe that $\square^d$ has $2d$ faces $\square_h^{d-1} = \partial\square_h^d$ for $h \in \{0, 1, \dots, 2d-1\}$ which are $\mathbb{R}^{d-1}$-cubes of side length 1. Thus, for each $h \in \{0, 1, \dots, 2d-1\}$, we can partition $\square_h^{d-1}$ into $2^{d-2}(d-1)!$ polytopes $\Delta_{h,\ell}^{d-1}$ such that $\Delta_{h,\ell}^{d-1}$ has d vertices for $\ell \in \{0, 1, \dots, 2^{d-2}(d-1)! - 1\}$. Then $\square^d$ can be partition into $2^{d-1}d!$ polytopes Δ_k^d for $k = \{1, 2, \dots, 2^{d-1}d!\}$ such that

  $$\Delta_k^d \ = \ \left\{\theta c + (1-\theta) \cdot y : \theta \in [0,1], y \in \Delta_{h,\ell}^{d-1}\right\}, \qquad k = h \cdot 2^{d-2}(d-1)! + \ell,$$

  with c being the center of $\square^d$.

The proof is complete. $\square$

3 Upper estimates on $\mathcal{N}_W^g(\varepsilon)$

In this section, we provide a quantitative study on the Hausdorff measure of $\mathcal{Z}_W^g$ for general continuous functions $g \in \mathcal{C}([0,1]^d, \mathbb{R}^m)$ and $W \subseteq \mathbb{R}^m$ being a global $\mathcal{C}^1$ manifold with $\dim(W) = p$. More precisely, recalling the definition of $\mathcal{Z}_W^h$ in (1.1), we establish an upper bound for

$$\mathcal{N}_W^g(\varepsilon) \ := \inf_{h \in \mathcal{C}([0,1]^d, \mathbb{R}^m), \|h-g\|_{\mathcal{C}^0} \leq \varepsilon} \mathcal{H}^{d+p-m}\left(\mathcal{Z}_W^h\right). \tag{3.10}$$

We shall assume that there exists a $\mathcal{C}^1$ diffeomorphism $\varphi : B_m(W, r) \to \varphi(B_m(W, r)) \subseteq \mathbb{R}^m$ with $\varphi(W) \ \subseteq \ \mathbb{R}^p \times \{0\} \subseteq \mathbb{R}^p \times \mathbb{R}^{m-p}$ for some $r > 0$. Recalling (2.8), we denote by

$$\gamma_W \ := \ \frac{\lambda_1}{\lambda_2}, \qquad \ell(s) \ = \ \frac{1}{2\sqrt{d}} \cdot \Psi_g(\gamma_W \cdot s) \qquad \text{for all } s > 0 \tag{3.11}$$

with

$$0 \ < \ \lambda_1 \ := \ \inf_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|} \ \leq \ \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|} \ := \ \lambda_2 \ < \ \infty. \tag{3.12}$$

Remark 3.1 *When W is the graph of a $\mathcal{C}^1$ function $\phi : \mathbb{R}^p \to \mathbb{R}^{m-p}$ with $\|\phi\|_{C^1} < \infty$. One can choose $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ such that*

$$\varphi(x, y) \;=\; (x, y - \phi(x)) \qquad \text{for all } x \in \mathbb{R}^p, y \in \mathbb{R}^{m-p}.$$

In this case, a direct computation yields

$$\min\left\{\frac{1}{2}, \frac{1}{\sqrt{1 + 4\|\phi\|^2_{\mathcal{C}^1}}}\right\} \;\leq\; \lambda_1 \;\leq\; \lambda_2 \;\leq\; \sqrt{2 + 2\|\phi\|^2_{\mathcal{C}^1}}.$$

Introducing the constant which approximately measures the set $g^{-1}\left(B_m(\overline{W}, \varepsilon)\right) \subseteq [0,1]^d$ in terms of Komolgorov ε-entropy

$$0 \;\leq\; \Lambda_\varepsilon \;:=\; \min\left\{(4\ell(\varepsilon))^d \cdot 2^{\mathbf{H}_{\ell(\varepsilon)}\left(g^{-1}\left(B_d(\overline{W}, \varepsilon)\right)\big|\mathbb{R}^d\right)}, 1\right\}, \tag{3.13}$$

we prove the following result.

Theorem 3.2 *Assume that $p + d \geq m$. Then for every $\varepsilon > 0$ sufficiently small such that $\Psi_g\left(\gamma_W \cdot \varepsilon\right) \leq \sqrt{d}$, it holds*

$$\mathcal{N}^g_W(\varepsilon) \;\leq\; C\Lambda_\varepsilon \cdot \left(\frac{1}{\Psi_g\left(\gamma_W \cdot \varepsilon\right)}\right)^{m-p} \tag{3.14}$$

with the constant $C = 2^{d+m-p-1} d! d^{d+\frac{p-m}{2}}$.

Proof. The proof is divided into several steps:

1. Fix $0 < \varepsilon < \dfrac{\lambda_2}{\lambda_1 + \lambda_2} \cdot r$, for every $\delta > 0$ such that

$$\omega_g\left(\sqrt{d}\delta\right) \;\leq\; \frac{\lambda_1 \varepsilon}{\lambda_2} \qquad \text{and} \qquad \varepsilon + \omega_g(\delta) < r \tag{3.15}$$

we divide $[0,1]^d$ into $(K_\delta)^d$ closed cubes $\square_\iota$ of side length $\ell_\delta = \dfrac{1}{K_\delta} \leq \delta$ with $K_\delta = \left\lfloor \dfrac{1}{\delta} \right\rfloor + 1$ and set

$$\mathcal{I}_\delta \;=\; \left\{\iota \in \left\{1, \ldots, \left(K_\delta\right)^d\right\} : \mathrm{int}(\square_\iota) \cap g^{-1}\left(B_d(\overline{W}, \varepsilon)\right) \neq \emptyset\right\}, \qquad \mathcal{O}_\delta \;=\; \bigcup_{\iota \in \mathcal{I}_\delta} \square_\iota. \tag{3.16}$$

From (2.7) and $\varepsilon + \omega_g(\delta) < r$, one has

$$g(\square_\iota) \;\subseteq\; B_m(\overline{W}, \varepsilon) + B_m(0, \omega_g(\delta)) \;\subseteq\; B_m(W, r) \qquad \text{for all } \iota \in I_\delta.$$

and this implies that $g(\mathcal{O}_\delta) \subseteq B_m(W, r)$. Therefore, one can define the function composition $\tilde{g} : \mathcal{O}_\delta \to \mathbb{R}^m$ such that

$$\tilde{g}(x) \;=\; \varphi(g(x)) \qquad \text{for all } x \in \mathcal{O}_\delta. \tag{3.17}$$

Since $\mathrm{dist}(g(x), W) \geq \varepsilon$ for all $x \in \partial O_\delta \backslash \partial [0,1]^d$, it holds

$$\inf_{x \in \partial O_\delta \backslash \partial [0,L]^d} \mathrm{dist}\left(\tilde{g}(x), \varphi(W)\right) \;\geq\; \inf_{|x-y| \geq \varepsilon} \left|\varphi(x) - \varphi(y)\right| \;\geq\; \lambda_1 \varepsilon. \tag{3.18}$$

In the next two steps, we will approximate $\tilde{g}$ by a function $\tilde{h}_\delta : \mathcal{O}_\delta \to \mathbb{R}^m$ such that (i). $\tilde{h}_\delta$ is a piecewise continuous function with

$$\left\|\tilde{h}_\delta - \tilde{g}\right\|_\infty \leq \lambda_2 \cdot \omega_g\left(\sqrt{d}\ell_\delta\right), \qquad \inf_{x\in\partial\mathcal{O}_\delta\backslash\partial[0,1]^d} \operatorname{dist}\left(\tilde{h}_\delta(x), \varphi(W)\right) \ > \ 0; \tag{3.19}$$

(ii). The $(d+p-m)$-Hausdorff measure of $\mathcal{Z}^{\tilde{h}_\delta}_{\varphi(W)} = \left\{x \in \mathcal{O}_\delta : \tilde{h}_\delta(x) \in \varphi(W)\right\}$ is bounded by

$$\mathcal{H}^{d+p-m}\left(\mathcal{Z}^{\tilde{h}_\delta}_{\varphi(W)}\right) \ \leq \ \left(2^{2d-1}d!d^{d+p-m}\right) \cdot \ell_\delta^{d+p-m} \cdot 2^{\mathbf{H}_{\ell_\delta}\left(g^{-1}\left(B_d(\overline{W},\varepsilon)\right)\Big|\mathbb{R}^d\right)}. \tag{3.20}$$

2. For every $\iota \in I_\delta$, following the induction process in Lemma 2.3, we partition $\square_\iota$ into $2^{d-1}d!$ polytopes Δ^k_ι in $\mathbb{R}^d$ such that the set of vertices $\mathbf{V}^k_\iota$ of Δ^k_ι has $(d+1)$ elements and is written by

$$\mathbf{V}^k_\iota \ = \ \left\{v^{k,j}_\iota \in \mathbb{R}^d : j \in \{1,2,\ldots,d+1\}\right\} \quad \text{for all } k \in \left\{1,2,\ldots,2^{d-1}d!\right\}.$$

Set $m_d := \min\{d,m\}$. Observe that for any given $\iota \in I_\delta$, $k \in \left\{1,2,\ldots,2^{d-1}d!\right\}$ and $s > 0$, there are m_d linearly independent vectors $z_1, z_2 \ldots, z_{m_d}$ in $\mathbb{R}^m$ with $\left|z_j - \tilde{g}\left(v^{k,j}_\iota\right)\right| < s$ for $j \in \{1,2,\ldots,m_d\}$ such that the following subspace of $\mathbb{R}^m$ has dimension $m_d + p - m$

$$\operatorname{span}\left\{z_1 - \tilde{g}\big(v^{k,d+1}_\iota\big), z_2 - \tilde{g}\big(v^{k,d+1}_\iota\big) \ldots, z_{m+d} - \tilde{g}\big(v^{k,d+1}_\iota\big)\right\} \bigcap \mathbb{R}^p \times \{0\}.$$

Thus, up to an arbitrary small modification on $\tilde{g}\left(v^{k,j}_\iota\right)$, we can assume that for every $\iota \in I_\delta$, $k \in \left\{1,2,\ldots,2^{d-1}d!\right\}$, the subspace

$$\operatorname{span}\left\{\tilde{g}\big(v^{k,1}_\iota\big) - \tilde{g}\big(v^{k,d+1}_\iota\big), \tilde{g}\big(v^{k,2}_\iota\big) - \tilde{g}\big(v^{k,d+1}_\iota\big) \ldots, \tilde{g}\big(v^{k,m_d}_\iota\big) - \tilde{g}\big(v^{k,d+1}_\iota\big)\right\} \bigcap \mathbb{R}^p \times \{0\} \tag{3.21}$$

has dimension $m_d + p - m$. Denote by $\nabla^d = \left\{\alpha \in \mathbb{R}^d : \alpha_j \geq 0, \sum_{j=1}^d \alpha_j \leq 1\right\}$, we have

$$\Delta^k_\iota \ = \ \left\{\sum_{j=1}^d \alpha_j \cdot v^{k,j}_\iota + \left(1 - \sum_{j=1}^d \alpha_j\right) \cdot v^{k,d+1}_\iota : \alpha \in \nabla^d\right\}.$$

The piecewise linear continuous function $\tilde{h}_\iota : \square_\iota \to \mathbb{R}^m$ is then defined as follows: for all $k \in \left\{1,2,\ldots,2^{d-1}d!\right\}$, $x = \sum_{j=1}^d \alpha_j \cdot v^{k,j}_\iota + \left(1 - \sum_{j=1}^d \alpha_j\right) \cdot v^{k,d+1}_\iota$ with $\alpha \in \nabla^d$, we set

$$\tilde{h}_\iota(x) \ := \ \tilde{g}\left(v^{k,d+1}_\iota\right) + \sum_{j=1}^d \alpha_j \cdot \left[\tilde{g}\left(v^{k,j}_\iota\right) - \tilde{g}\left(v^{k,d+1}_\iota\right)\right]. \tag{3.22}$$

From (3.17) and (3.12), one estimates

$$\begin{aligned}\left|\tilde{h}_\iota(x) - \tilde{g}(x)\right| \ &\leq \ \sup_{|y-z|\leq \operatorname{diam}\left(\Delta^k_\iota\right)} |\tilde{g}(y) - \tilde{g}(z)| \\ &\leq \ \lambda_2 \cdot \sup_{|y-z|\leq\sqrt{d}\ell_\delta} |g(y) - g(z)| \ \leq \ \lambda_2 \cdot \omega_g\left(\sqrt{d}\ell_\delta\right).\end{aligned}$$

The function $\tilde{h}_\delta : \mathcal{O}_\delta \to \mathbb{R}^m$ is defined by

$$\tilde{h}_\delta(x) \ = \ \tilde{h}_\iota(x) \qquad \text{for all } x \in \square_\iota, \iota \in I_\delta$$

is continuous and satisfies

$$\left|\tilde{h}_\delta(x) - \tilde{g}(x)\right| \ \leq \ \lambda_2 \cdot \omega_g\left(\sqrt{d}\ell_\delta\right) \quad \text{for all } x \in \mathcal{O}_\delta.$$

Recalling (3.18) and (3.15), we have

$$\inf_{x\in\partial\mathcal{O}_\delta\backslash\partial[0,L]^d} \operatorname{dist}\left(\tilde{h}_\delta(x), \varphi(W)\right) \ \geq \ \lambda_1\varepsilon - \lambda_2 \cdot \omega_g\left(\sqrt{d}\ell_\delta\right) \ > \ 0. \tag{3.23}$$

3. Let us show that $\tilde{h}_\delta$ satisfies (ii). Fix $\iota \in I_\delta$, we consider the $m \times d$ matrix

$$\mathbf{A}_\iota^k = \left[\tilde{g}\left(v_\iota^{k,1}\right) - \tilde{g}\left(v_\iota^{k,d+1}\right), \ldots, \tilde{g}\left(v_\iota^{k,d}\right) - \tilde{g}\left(v_\iota^{k,d+1}\right)\right].$$

By the rank-nullity theorem, one has that $\operatorname{rank}(\mathbf{A}_\iota^k) = m_d$ and

$$\left\{\alpha \in \mathbb{R}^d : \mathbf{A}_\iota^k\alpha = 0\right\} \ = \ \mathbf{Y}_\iota^k \qquad \text{with} \qquad \dim\left(\mathbf{Y}_\iota^k\right) \ = \ d - m_d.$$

Assume that $\mathbf{X}_\iota^k \oplus \mathbf{Y}_\iota^k = \mathbb{R}^d$. The linear map $\alpha \mapsto \mathbf{A}_\iota^k\alpha$ is injective from $\mathbf{X}_\iota^k$ to $\mathbb{R}^m$ and $\dim\left(\mathbf{X}_\iota^k\right) = m_d$. Thus, from (3.21), the following set is a (m_d+p-m)-dimensional hyperplane

$$\Gamma_\iota^k \ := \ \left\{\alpha \in \mathbf{X}_\iota^k : \mathbf{A}_\iota^k\alpha \in \mathbb{R}^p \times \{0\} - \tilde{g}\left(v_\iota^{k,d+1}\right) \subset \mathbb{R}^p \times \mathbb{R}^{m-p}\right\}.$$

For every $k \in \left\{1, 2, \ldots, 2^{d-1}d!\right\}$, we set

$$\nabla_k^d \ := \ \left\{\alpha \in \nabla^d : \tilde{h}_\iota\left(\sum_{j=1}^d \alpha_j \cdot v_\iota^{k,j} + \left(1 - \sum_{j=1}^d \alpha_j\right) \cdot v_\iota^{k,d+1}\right) \in \varphi(W) \subset \mathbb{R}^p \times \{0\}\right\}.$$

From (3.22), it holds

$$\begin{aligned}
\nabla_k^d \ &= \ \left\{\alpha \in \nabla^d : \tilde{g}\left(v_\iota^{k,d+1}\right) + \sum_{j=1}^d \alpha_j \cdot \left[\tilde{g}\left(v_\iota^{k,j}\right) - \tilde{g}\left(v_\iota^{k,d+1}\right)\right] \in \varphi(W)\right\} \\
&= \ \left\{\alpha \in \nabla^d : \sum_{j=1}^d \alpha_j \cdot \left[\tilde{g}\left(v_\iota^{k,j}\right) - \tilde{g}\left(v_\iota^{k,d+1}\right)\right] \in \varphi(W) - \tilde{g}\left(v_\iota^{k,d+1}\right)\right\} \\
&\subseteq \ \left\{\alpha \in \mathbb{R}^d : \mathbf{A}_\iota^k\alpha \in \mathbb{R}^p \times \{0\} - \tilde{g}\left(v_\iota^{k,d+1}\right)\right\} \ = \ \mathbf{Y}_\iota^k + \Gamma_\iota^k.
\end{aligned}$$

Observe that $\mathbf{Y}_\iota^k + \Gamma_\iota^k$ is a $(d + p - m)$-dimensional hyperplane. Again recalling (3.22), we obtain

$$\left\{x \in \Delta_\iota^k : \tilde{h}_\iota(x) \in \varphi(W)\right\} \ = \ \left\{\sum_{j=1}^d \alpha_j \cdot v_\iota^{k,j} + \left(1 - \sum_{j=1}^d \alpha_j\right) \cdot v_\iota^{k,d+1} : \alpha \in \nabla_k^d\right\}$$

, and

$$\begin{aligned}\mathcal{H}^{d-m+p}\left(\left\{x\in\Delta_\iota^k:\tilde{h}_\iota(x)\in\varphi(W)\right\}\right) &\le \left(\sup_{j\in\{1,\dots,d\}}\left|v_\iota^{k,j}-v_\iota^{k,d+1}\right|\right)^{d-m+p}\cdot\mathcal{H}^{d-m+p}\left(\nabla_k^d\right)\\ &\le \left(\sqrt{d}\ell_\delta\right)^{d-m+p}\cdot\mathcal{H}^{d-m+p}\left(\nabla_k^d\right) \le (d\ell_\delta)^{d-m+p}.\end{aligned}$$

Thus, for all $\iota\in\mathcal{I}_\delta$, it holds

$$\mathcal{H}^{d-m+p}\left(\mathcal{Z}_{\varphi(W)}^{\tilde{h}_\iota}\right) \le \sum_{k=1}^{2^{d-1}d!}\mathcal{H}^{d-m+p}\left(\left\{x\in\Delta_\iota^k:\tilde{h}_\iota(x)\in\varphi(W)\right\}\right) \le 2^{d-1}d!(d\ell_\delta)^{d-m+p}.$$

By the concept of ε-entropy in Definition 2.2, we have

$$g^{-1}\left(B_m(\overline{W},\varepsilon)\right) \subseteq \bigcup_{j=1}^{J}D_j,\qquad J \le 2^{\mathbf{H}_{\ell_\delta}\left(g^{-1}\left(B_m(\overline{W},\varepsilon)\right)\middle|\mathbb{R}^d\right)},$$

for some $D_j\subset\mathbb{R}^d$ with $\mathrm{diam}(D_j)\le 2\ell_\delta$. For any $j\in\{1,\dots,J\}$, it holds

$$\#\{\iota\in\left\{1,\dots,\left(K_\delta\right)^d\right\}:\mathrm{int}(\square_\iota)\cap D_j\neq\emptyset\} \le 2^d.$$

Hence,

$$\#\left(\mathcal{I}_\delta\right) \le \min\left\{2^d\cdot 2^{\mathbf{H}_{\ell_\delta}\left(g^{-1}\left(B_m(\overline{W},\varepsilon)\right)\middle|\mathbb{R}^d\right)},\left(K_\delta\right)^d=\left(\frac{1}{\ell_\delta}\right)^d\right\}, \tag{3.24}$$

and this yields (3.20) by

$$\mathcal{H}^{d+p-m}\left(\mathcal{Z}_{\varphi(W)}^{\tilde{h}_\delta}\right) \le \sum_{\iota\in\mathcal{I}_\delta}\mathcal{H}^{d-m+p}\left(\mathcal{Z}_{\varphi(W)}^{\tilde{h}_\iota}\right) \le \#\left(\mathcal{I}_\delta\right)\cdot 2^{d-1}d!(d\ell_\delta)^{d-m+p}. \tag{3.25}$$

4. To complete the proof, we first approximate g by the continuous function $g_\delta:\mathcal{O}_\delta\mapsto\mathbb{R}^m$ which is defined by

$$g_\delta(x) = \varphi^{-1}(\tilde{h}_\delta(x))\qquad\text{for all } x\in\mathcal{O}_\delta.$$

From (3.19), it holds

$$|g_\delta(x)-g(x)| = \left|\varphi^{-1}(\tilde{h}_\delta(x))-\varphi^{-1}(\tilde{g}(x))\right| \le \frac{|\tilde{h}_\delta(x)-\tilde{g}(x)|}{\lambda_1} \le \frac{\lambda_2}{\lambda_1}\cdot\omega_g\left(\sqrt{d}\ell_\delta\right),$$

and

$$\begin{aligned}\inf_{\partial\mathcal{O}_\delta\backslash\partial[0,L]^d}\mathrm{dist}(g_\delta(x),W) &= \inf_{\partial\mathcal{O}_\delta\backslash\partial[0,1]^d}\mathrm{dist}\big(\varphi^{-1}(\tilde{h}_\delta(x)),\varphi^{-1}(\varphi(W))\big)\\ &\ge \frac{1}{\lambda_2}\cdot\mathrm{dist}\left(\tilde{h}_\delta(x),\varphi(W)\right) > 0.\end{aligned}$$

Since $\mathrm{dist}\left(g(x),W\right)\ge\varepsilon$ for all $x\in\left([0,1]^d\backslash\mathcal{O}_\delta\right)$, we can extend g_δ to $[0,1]^d$ such that g_δ is still continuous with $\|g_\delta-g\|_\infty\le\frac{\lambda_2}{\lambda_1}\cdot\omega_g\left(\sqrt{d}\ell_\delta\right)$ and $g_\delta(x)$ does not belong to W for every

$x \in [0,1]^d \backslash \mathcal{O}_\delta$. Thus, (3.25) and (3.24) yield

$$\begin{aligned}
\mathcal{H}^{d+p-m}\left(\mathcal{Z}_W^{g_\delta}\right) &= \mathcal{H}^{d+p-m}\left(\{x \in \mathcal{O}_\delta : g_\delta(x) \in W\}\right) \\
&= \mathcal{H}^{d+p-m}\left(\left\{x \in \mathcal{O}_\delta : \tilde{h}_\delta \in \varphi(W)\right\}\right) \;=\; \mathcal{H}^{d+p-m}\left(\mathcal{Z}_{\varphi(W)}^{\tilde{h}_\delta}\right) \\
&\leq \frac{2^{d-1}d!d^{d-m+p}}{\ell_\delta^{m-p}} \cdot \min\left\{2^d\ell_\delta^d \cdot 2^{\mathbf{H}_{\ell_\delta}\left(g^{-1}\left(B_d(\overline{W},\varepsilon)\right)\Big|\mathbb{R}^d\right)}, 1\right\}.
\end{aligned}$$

Recalling (2.7) and (2.8), we choose $\delta = \frac{1}{\sqrt{d}} \cdot \Psi_g\left(\frac{\lambda_1\varepsilon}{\lambda_2}\right)$ such that $\omega_g\left(\sqrt{d}\delta\right) \leq \frac{\lambda_1\varepsilon}{\lambda_2}$, the condition (3.15) on δ holds, and

$$\|g_\delta - g\|_\infty \;\leq\; \frac{\lambda_2}{\lambda_1} \cdot \omega_g\left(\sqrt{d}\ell_\delta\right) \;\leq\; \varepsilon,$$

and

$$\frac{1}{2\sqrt{d}} \cdot \Psi_g\left(\frac{\lambda_1\varepsilon}{\lambda_2}\right) \;=\; \frac{\delta}{2} \;\leq\; \ell_\delta \;=\; \frac{1}{\left\lfloor \frac{1}{\delta} \right\rfloor + 1} \;\leq\; \delta \;=\; \frac{1}{\sqrt{d}} \cdot \Psi_g\left(\frac{\lambda_1\varepsilon}{\lambda_2}\right).$$

Hence, (3.11)-(3.13) yields (3.14) and the proof is complete. □

Remark 3.3 *In addition, if $g \in \mathcal{C}^\alpha([0,1]^d, \mathbb{R}^m)$ is Hölder continuous with exponent $\alpha \in (0,1]$ then from (2.9) it holds*

$$\Psi_g\left(\gamma_W \cdot \varepsilon\right) \;\geq\; \left(\frac{\gamma_W \cdot \varepsilon}{\|g\|_{\mathcal{C}^{0,\alpha}}}\right)^{\frac{1}{\alpha}} \qquad \text{for all } \varepsilon \geq 0.$$

Recalling (3.13)-(3.14), we obtain Theorem 1.1 with $C_W = \left(2^{d+m-p-1}d^{d+\frac{p-m}{2}}d!\right) \cdot \left(\frac{1}{\gamma_W}\right)^{\frac{m-p}{\alpha}}$.

To conclude this section, let us provide an example to show that the blow up rate $\left(\frac{1}{\varepsilon}\right)^{\frac{m-p}{\alpha}}$ with respect to ε is the best bound in terms of power function in the case $d = m = 1, p = 0$, $W = \{0\}$, and $\alpha = 1$.

Example 3.4 *Consider the Lipschitz function $g : [0,1] \to \mathbb{R}$ with the Lipscthiz constant 1 such that*

$$g(x) \;=\; \sum_{n=1}^{\infty} u_n(x) \cdot \chi_{[s_n, s_{n+1}]}(x) \quad \text{with} \quad s_1 \;=\; 0, s_n = \sum_{j=1}^{n-1} 2^{-j} \quad \text{for all } n \geq 2.$$

Here, the function $u_n : [0,1] \to \mathbb{R}$ is defined as follows: $u_n = 0$ on $[0,1]\backslash(s_n, s_{n+1})$ and for all $x \in [s_n, s_{n+1}]$

$$u_n(x) \;=\; 2^{-(n^2+n)} \cdot \sum_{k=0}^{2^{n^2}-1} u\left(2^{n^2+n} \cdot \left[(x - s_n) - k2^{-(n^2+n)}\right]\right) \cdot \chi_{[s_n + k2^{-(n^2+n)}, s_n + (k+1)2^{-(n^2+n)}]}$$

with

$$u(x) \;=\; \left(\frac{1}{4}-\left|x-\frac{1}{4}\right|\right)\cdot\chi_{[0,1/2]}+\left(\left|x-\frac{3}{4}\right|-\frac{1}{4}\right)\cdot\chi_{[1/2,1]}.$$

Given any $\varepsilon \in \left[2^{-(n+1)^2-(n+1)}, 2^{-(n^2+n)}\right[$*, for any* $h \in C([0,1],\mathbb{R})$ *with* $\|h-g\|_{\mathcal{C}^0([0,1])} \le \varepsilon$*, we have*

$$\mathcal{H}^0\left(\mathcal{Z}^h_{\{0\}}\right) \;=\; \#\{x\in[0,1] : h(x)=0\} \;\ge\; \#\{x \in]s_n, s_{n+1}[: u_n(x)=0\} \;\ge\; 2^{n^2} \;\ge\; \left(\frac{1}{\varepsilon}\right)^{\frac{1}{1+o(\varepsilon)}}$$

with $\lim_{\varepsilon\to 0+} o(\varepsilon)=0$*. Thus,*

$$\mathcal{N}^g_{\{0\}}(\varepsilon) \;=\; \inf_{\|h-g\|_{\mathcal{C}^0}\le\varepsilon} \mathcal{H}^0\left(\mathcal{Z}^h_{\{0\}}\right) \;\ge\; \left(\frac{1}{\varepsilon}\right)^{\frac{1}{1+o(\varepsilon)}},$$

and the blow up rate $\left(\frac{1}{\varepsilon}\right)^{\frac{m-p}{\alpha}} = \frac{1}{\varepsilon}$ *in Theorem 1.1 is optimal in terms of power function in the case* $d=m=1, p=0$ *and* $\alpha=1$*. In this scalar case, one can follow the same construction to show that the rate is optimal for* $\alpha\in(0,1)$*.*

For the multi-dimensional cases ($d\ge 2$), the blow up rate $\left(\frac{1}{\varepsilon}\right)^{\frac{m-p}{\alpha}}$ in Theorem 1.1 should be still optimal in terms of power function but the situation becomes considerably more technical. We leave this open.

4 A quantitative bound on the total number of shock curves

In this section, we shall use Theorem 3.2 to prove Theorem 1.2. In general, the scalar conservation laws (1.3) do not possess classical solutions since discontinuities arise in finite time even if the initial data are smooth. Hence, it is natural to consider weak solutions in the sense of distributions that, for sake of uniqueness, satisfy an entropy criterion for admissibility

$$u(t,x-) \;\ge\; u(t,x+) \qquad \text{for } a.e.\ t>0, x\in\mathbb{R}.$$

Under the convexity assumption (1.4), it is well known (see e.g. in [8]) that for every $\bar{u} \in \mathbf{L}^\infty(\mathbb{R})\cap\mathbf{L}^1(\mathbb{R})$, the Cauchy problem (1.3) with $u(0,\cdot)=\bar{u}$ admits a unique entropy solution $u(t,x)$ which satisfies the Oleinik's estimate

$$u(t,y)-u(t,x) \;\le\; \frac{1}{\lambda t}\cdot(y-x) \qquad \text{for all } t>0, y>x.$$

Moreover, the solution is continuous except on the union of an at most countable set of Lipschitz continuous curves (shocks). To be precise, we recall the definition and theory of generalized characteristic curves associated to (1.3). For a more in depth theory of generalized characteristics, we direct the readers to [10].

Definition 4.1 *A Lipscthiz continuous curve* $\xi(t)$ *defined on an interval* $[0,\infty)$ *is called a generalized characteristic if for a.e.* t *in the interval*

$$\dot{\xi}(s) \;\in\; \left[f'(u(s,\xi(s)+)), f'(u(s,\xi(s)-))\right]. \tag{4.26}$$

Moreover, we say that

- *ξ on $[a,b]$ is genuine if $u(s,\xi(s)+) = u(s,\xi(s)-)$ for a.e. $t \in [a,b]$.*
- *ξ on an interval $[\bar{t},\sigma)$ for some $\bar{t} < \sigma \le +\infty$ is a shock if*

$$u(t,\xi(t)-) \;>\; u(t,\xi(t)+) \qquad \text{for all } t \in [\bar{t},\sigma).$$

- *A point $(\bar{t},\bar{x}) \in (0,\infty)\times\mathbb{R}$ is called a shock generation point if the forward characteristic through $(\bar{t},\bar{x})$ is a shock, while every backward characteristic through $(\bar{t},\bar{x})$ is genuine.*

The existence of backward (forward) characteristics was studied by Fillipov. As in [10], the speed of the characteristic curves are determined and genuine characteristics are essentially classical characteristics.

Proposition 4.1 *Let $\xi : [a,b] \to \mathbb{R}$ be a generalized characteristic curve of (1.3), associated with an entropy weak solution u. Then for almost every time $t \in [a,b]$, it holds that*

$$\dot{\xi}(t) \;=\; \begin{cases} f'(u(t,\xi(t))) & \text{if} \qquad u(t,\xi(t)+) \;=\; u(t,\xi(t)-)\,, \\ \dfrac{f(u(t,\xi(t)+)) - f(u(t,\xi(t)-))}{u(t,\xi(t)+) - u(t,\xi(t)-)} & \text{if} \qquad u(t,\xi(t)+) \;<\; u(t,\xi(t)-)\,. \end{cases} \tag{4.27}$$

In addition, if ξ is genuine on $[a,b]$, then $(t,\xi(\cdot))$ is a straight line and the solution u is constant along this line.

Given $(t,x) \in (0+\infty)\times\mathbb{R}$, all backward characteristics ξ are confined between a maximal and minimal backward characteristics, denoted by $\xi^+_{(t,x)}$ and $\xi^-_{(t,x)}$. Moreover, we recall properties of generalized characteristics, including the non-crossing property of two genuine characteristics.

Proposition 4.2 *Let u be an entropy weak solution to* (1.3)*. Then for any $(t,x) \in]0,+\infty[\times\mathbb{R}$, the followings hold:*

(i) The maximal and minimal backward characteristics $\xi^{\pm}_{(t,x)}$ are genuine.

(ii) There is a unique forward characteristic, denoted by $\xi^{(t,x)}$, which passes though (t,x). If $u(t,\cdot)$ is discontinuous at a point x, then

$$u\left(\tau, \xi^{(t,x)}(\tau)-\right) \;>\; u\left(\tau, \xi^{(t,x)}(\tau)+\right) \qquad \text{for all } \tau \ge t\,.$$

(iii) Two genuine characteristics may intersect only at their endpoints.

From the above proposition, one can easily obtain the following lemma.

Lemma 4.2 *For any given initial data $\bar{v} \in \mathcal{C}^2(\mathbb{R})$ with supp$(\bar{v}) \subseteq [-R,R]$ such that*

$$\#\left\{x \in [-R,R] : [f'(\bar{v})(x)]'' = 0\right\} \;<\; \infty \tag{4.28}$$

The total number of shock curves of the entropy weak solution v of (1.3) with $u(0,\cdot) = \bar{v}$ is at most the total number of inflection points of $f'(\bar{v})$.

Proof. From Proposition 4.2, we have that the total number of shock curves of v is bounded by the total number of shock generation points. Given a shock generation point $(\bar{t},\bar{x})$, let $\mathbf{d}:\mathbb{R}\to\mathbb{R}$ be a $\mathcal{C}^2$ such that

$$\mathbf{d}(\beta) \;=\; \beta + f'(\bar{v}(\beta))\cdot\bar{t} \qquad \text{for all } \beta\in\mathbb{R}.$$

Two cases are considered:

- If $v(\bar{t},\bar{x}-) = v(\bar{t},\bar{x}+)$ then let $\xi_{(\bar{t},\bar{x})}(\cdot)$ be the backward characteristic starting from $(\bar{t},\bar{x})$. Set $\bar{\beta} := \xi_{(\bar{t},\bar{x})}(0)$. From ([10, Lemma 5.2]), it holds

$$\mathbf{d}'(\bar{\beta}) \;=\; 0 \qquad \Longrightarrow \qquad [f'(\bar{v})]'(\bar{\beta}) \;=\; -\frac{1}{\bar{t}}. \tag{4.29}$$

  For every $\delta>0$, there exist $\bar{x}-\delta < x_\delta^- < \bar{x} < x_\delta^+ < \bar{x}+\delta$ such that $v(\bar{t},\cdot)$ is continuous at $x_\delta^\pm$. By the non-crossing property (iii) in Proposition 4.2 and the continuity of $\bar{v}(\bar{t},\cdot)$ at $\bar{x}$, we have

$$\xi_{(\bar{t},x_\delta^-)}(0) \;:=\; \beta_\delta^- \;<\; \bar{\beta} \;<\; \beta_\delta^- \;:=\; \xi_{(\bar{t},x_\delta^+)}(0), \qquad \lim_{\delta\to 0+}\beta_\delta^- \;=\; \lim_{\delta\to 0+}\beta_\delta^+ \;=\; \bar{\beta}.$$

  and

$$x_\delta^- \;=\; \mathbf{d}(\beta_\delta^-) \;<\; \mathbf{d}(\bar{\beta}) \;=\; \bar{x} \;<\; \mathbf{d}(\beta_\delta^+) \;=\; x_\delta^+.$$

  This implies that there exist $\tilde{\beta}_\delta^- \in \left(\beta_\delta^-,\bar{\beta}\right)$ and $\tilde{\beta}_\delta^+ \in \left(\bar{\beta},\beta_\delta^+\right)$ such that

$$\mathbf{d}'\left(\tilde{\beta}_\delta^\pm\right) \;>\; 0 \qquad \Longrightarrow \qquad [f'(\bar{v})]'\left(\tilde{\beta}_\delta^\pm\right) \;>\; -\frac{1}{\bar{t}}.$$

  Hence, (4.29) and the assumption (4.28) imply that $\bar{\beta}$ is an inflection point of $f'(\bar{v})$.

- Otherwise, if $v(\bar{t},\bar{x}-) > v(\bar{t},\bar{x}+)$ then $(\bar{t},\bar{x})$ is a center of a centered compression wave, i.e., there are two genuine backward characteristic ξ_1 and ξ_2 through $(\bar{t},\bar{x})$ so that every backward characteristic through $(\bar{t},\bar{x})$ contained in the funnel confined between ξ_1 and ξ_2. In this case, one has that

$$[f'(v)]'(\beta) \;=\; -\frac{1}{\bar{t}} \qquad \text{for all } \beta\in(\xi_1(0),\xi_2(0))$$

  and this contradicts to (4.28).

Therefore, the total number of shock generation points is at most the total number of inflection points of $f'(\bar{v})$. □

From the above lemma and Theorem 3.2, we now extend Theorem 1.2 to the case of $\mathcal{C}^3$-smooth f. In order to do so, given constants $R,V>0$, we denote by

$$\Phi_{f,V,R}(\varepsilon) \;=\; 2^{12}\cdot\max\left\{45\left(1+\frac{1}{V}\right)\|f\|_{C^3(-\frac{V}{2},\frac{V}{2})},\ \frac{4\beta_\varepsilon}{\Psi^V_{f^{(3)}_{(-\frac{V}{2},\frac{V}{2})}}(\beta_\varepsilon)}\right\} \tag{4.30}$$

with $\beta_\varepsilon = \dfrac{5\lambda\varepsilon^3}{2^9V^4R^3}$ and $\Psi^V_{f^{(3)}}$ being the inverse of the minimal modulus of a continuity of the restriction of $f^{(3)}$ to the interval $(-\frac{V}{2},\frac{V}{2})$ which is defined in (2.8).

Theorem 4.3 *Given constants $R, V > 0$, assume that $f \in \mathcal{C}^3(\mathbb{R})$ and $\bar{u} \in \mathbf{L}^1(\mathbb{R})$ satisfies (1.5). Then, for every $\varepsilon > 0$ sufficiently small, there exists $\bar{v} \in \mathcal{C}^2(\mathbb{R})$ with $\mathrm{Supp}(\bar{v}) \subseteq [-2R, 2R]$ and $\|\bar{v} - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon$, such that the entropy weak solution $v = v(t, x)$ of (1.3) with initial datum $v(0, \cdot) = \bar{v}$ satisfies*

$$[\text{Total number of shock curves of } v] \ \leq \ \frac{\Phi_{f,V,R}(\varepsilon)}{\lambda} \cdot \frac{R^4 V^5}{\varepsilon^4} + 4. \tag{4.31}$$

Proof. 1. Let $\bar{u} \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$ be such

$$\mathrm{Supp}(\bar{u}) \subseteq [-R, R] \qquad \text{and} \qquad \mathrm{Tot.Var.}(\bar{u}, (-\infty, \infty)) \ \leq \ V.$$

For every $\delta > 0$, we first approximate $\bar{u}$ by the smooth function $u_\delta \in \mathcal{C}^3(\mathbb{R})$ with $\mathrm{Supp}(u_\delta) \subseteq [-R - \delta, R + \delta]$ which is defined by

$$u_\delta(x) \ := \ [\bar{u} * \rho_\delta](x) \ = \ \int_{-\infty}^{\infty} \bar{u}(y)\rho_\delta(x - y)dy \quad \text{for all } x \in \mathbb{R}$$

where the mollifier

$$\rho_\delta(x) = \frac{315}{256 \cdot \delta} \cdot \left(1 - \frac{x^2}{\delta^2}\right)^4 \cdot \chi_{[-\delta,\delta]}(x)$$

is a $\mathcal{C}^4(\mathbb{R})$ function with $\mathrm{Supp}(\rho_\delta) \subseteq [-\delta, \delta]$ and $\displaystyle\int_{-\infty}^{\infty} \rho_\delta(x)dx = 1$. From [1, Lemma 3.24], the $\mathbf{L}^1$-distance between $\bar{u}$ and u_δ is bounded by

$$\|u_\delta - \bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \ \leq \ \delta \cdot \mathrm{Tot.Var.}(\bar{u}, (-\infty, \infty)) \ \leq \ V\delta. \tag{4.32}$$

Moreover, a direct computation yields

$$\|u_\delta\|_{\mathbf{L}^\infty(\mathbb{R})} \ \leq \ \|u\|_{\mathbf{L}^\infty(\mathbb{R})} \cdot \|\rho_\delta\|_{\mathbf{L}^1(\mathbb{R})} \ = \ \|u\|_{\mathbf{L}^\infty(\mathbb{R})} \ \leq \ \frac{1}{2} \cdot \mathrm{Tot.Var.}(\bar{u}, (-\infty, \infty)) \ \leq \ \frac{V}{2},$$

$$\left\|u'_\delta\right\|_{\mathbf{L}^\infty(\mathbb{R})} \ \leq \ \|u\|_{\mathbf{L}^\infty(\mathbb{R})} \cdot \left\|\rho'_\delta\right\|_{\mathbf{L}^1(\mathbb{R})} \ \leq \ \frac{V}{2} \cdot \left\|\rho'_\delta\right\|_{\mathbf{L}^1(\mathbb{R})} \ = \ \frac{315V}{256 \cdot \delta},$$

and

$$\left\|u''_\delta\right\|_{\mathbf{L}^\infty(\mathbb{R})} \ \leq \ \frac{V}{2} \cdot \left\|\rho''_\delta\right\|_{\mathbf{L}^1(\mathbb{R})} \ = \ \frac{1215V}{98\sqrt{7} \cdot \delta^2}, \qquad \left\|u'''_\delta\right\|_{\mathbf{L}^\infty(\mathbb{R})} \ \leq \ \frac{V}{2} \cdot \left\|\rho'''_\delta\right\|_{\mathbf{L}^1(\mathbb{R})} \ < \ \frac{5085V}{224 \cdot \delta^3}.$$

Consider the continuous function $h_\delta := [f'(u_\delta)]''$ with $\mathrm{Supp}(h_\delta) \subseteq [-R-2\delta, R+2\delta]$. For every $x, y \in \mathbb{R}$, we can roughly estimate

$$\begin{aligned}
\left|h_\delta(y) - h_\delta(x)\right| \ &= \ \left|\left[f'''(u_\delta)[u'_\delta]^2 + f''(u_\delta)u''_\delta\right](y) - \left[f'''(u_\delta)[u'_\delta]^2 + f''(u_\delta)u''_\delta\right](x)\right| \\
&\leq \ \frac{45V(1+V)\left\|f\right\|_{C^3(-\frac{V}{2},\frac{V}{2})}}{2\delta^3} \cdot |x - y| + \left\|u'_\delta\right\|_\infty^2 \cdot \left|f^{(3)}(u_\delta(x)) - f^{(3)}(u_\delta(y))\right| \\
&\leq \ \frac{45V(1+V)\left\|f\right\|_{C^3(-\frac{V}{2},\frac{V}{2})}}{2\delta^3} \cdot |x - y| + \frac{8V^2}{5\delta^2} \cdot \omega_{f^{(3)}}\left(\frac{5V}{4\delta} \cdot |x - y|\right)
\end{aligned}$$

and (2.7) yields

$$\omega_{h_\delta}(\tau) \ \leq \ \frac{45V(1+V)\left\|f\right\|_{C^3(-\frac{V}{2},\frac{V}{2})}}{2\delta^3} \cdot \tau + \frac{8V^2}{5\delta^2} \cdot \omega_{f^{(3)}}\left(\frac{5V\tau}{4\delta}\right) \qquad \text{for all } \tau \geq 0.$$

Recalling (2.8), we then derive an upper bound on the inverse of the minimal modulus of a continuity of h_δ

$$\Psi_{h_\delta}(s) \ \geq \ \min\left\{\frac{\delta^3 s}{45V(V+1)\|f\|_{C^3(-\frac{V}{2},\frac{V}{2})}}, \frac{4\delta}{5V}\cdot\Psi^V_{f^{(3)}_{(-\frac{V}{2},\frac{V}{2})}}\left(\frac{5\delta^2 s}{16V^2}\right)\right\} \tag{4.33}$$

where $\Psi^V_{f^{(3)}}$ is the inverse of the minimal modulus of continuity of the restriction of $f^{(3)}$ on $(-V/2, V/2)$.

On the other hand, applying Theorem 3.2 for $m = d = 1, p = 0$, $W = \{0\} \in \mathbb{R}$, and $\Lambda_\varepsilon \leq 1$, we get that for any given $\sigma > 0$ sufficiently small, there exists a continuous function $\tilde{h}_{\sigma,\delta}$ such that

$$\mathrm{Supp}(\tilde{h}_{\sigma,\delta}) \subseteq (-R-2\delta, R+2\delta), \qquad \|\tilde{h}_{\sigma,\delta} - h_\delta\|_{\mathcal{C}^0(\mathbb{R})} \ \leq \ \sigma \tag{4.34}$$

and

$$\#\{x \in (-R-2\delta, R+2\delta) : \tilde{h}_{\sigma,\delta}(x) = 0\} \ \leq \ \frac{4(R+\delta)}{\Psi_{h_\delta}(\sigma)}. \tag{4.35}$$

2. Set $R_1 := \max\{R+\delta, \sup\{x \in \mathbb{R} : \tilde{h}_{\sigma,\delta}\} \neq 0\} \in (R+\delta, R+2\delta)$ and

$$\alpha_1 = \int_{-R-2\delta}^{R_1} \tilde{h}_{\sigma,\delta}(z)dz, \qquad \alpha_0 \ = \ \int_{-R-2\delta}^{R_1}\left(\int_{-R-2\delta}^{y} \tilde{h}_{\sigma,\delta}(z)dz\right)dy.$$

We approximate $f'(u_\delta)$ by a function $F_{\sigma,\delta}$ defined by

$$F_{\sigma,\delta}(x) \ = \ \begin{cases} f'(0), & x \in \mathbb{R}\backslash[-R-2\delta, R+2\delta], \\ f'(0) + \displaystyle\int_{-R-2\delta}^{x}\left(\int_{-R-2\delta}^{y}\tilde{h}_{\sigma,\delta}(z)dz\right)dy, & x \in (-R-2\delta, R_1), \\ f'(0) + \alpha_0 \cdot \left(\dfrac{R+2\delta-x}{R+2\delta-R_1}\right)^3 + G_\theta(x)\chi_{[R_1,R_1+\theta]}, & x \in [R_1, R+2\delta), \end{cases} \tag{4.36}$$

with

$$\begin{cases} G_\theta(x) \ = \ (x-R_1)(x-R_2)^3 \cdot \left(\dfrac{\alpha_2}{(R_1-R_2)^3} + (x-R_1)\cdot\left(\dfrac{\alpha_3}{2(R_1-R_2)^3} - \dfrac{3\alpha_2}{(R_1-R_2)^4}\right)\right), \\ \alpha_2 \ = \ \alpha_1 + \dfrac{3\beta_1}{R+2\delta-R_1}, \qquad \alpha_3 \ = \ \dfrac{-6\beta_1}{R+2\delta-R_1}, \qquad R_2 \ = \ R_1+\theta, \end{cases}$$

for some $\theta > 0$ sufficiently small. One computes that

$$G_\theta(R_1) \ = \ G_\theta(R_2) \ = \ G'_\theta(R_2) \ = \ G''_\theta(R_2) \ = \ 0, \quad G'_\theta(R_1) \ = \ \alpha_2, \quad G''_\theta(R_1) \ = \ \alpha_3,$$

and this yields

$$\begin{cases} F_{\sigma,\delta}(R_1\pm) \ = \ f'(0) + \alpha_0, \quad F'_{\sigma,\delta}(R_1\pm) \ = \ \alpha_1, \quad F''_{\sigma,\delta}(R_1\pm) \ = \ \tilde{h}_{\sigma,\delta}(R_1) \ = \ 0, \\ F_{\sigma,\delta}(R+2\delta) \ = \ F'_{\sigma,\delta}(R+2\delta) \ = \ F''_{\sigma,\delta}(R+2\delta) \ = \ 0. \end{cases}$$

Hence, $F_{\sigma,\delta}$ is a C^2-function. Moreover, observe that the number of inflection points of $F_{\sigma,\delta}$ in $[R_1, R+2\delta]$ is less than 5, we have from (4.35) that

$$\#\{x \in \mathbb{R} : x \text{ is an inflection point of } F_{\sigma,\delta}\} \ \leq \ \frac{4(R+\delta)}{\Psi_{h_\delta}(\sigma)} + 4. \tag{4.37}$$

Recalling (4.34), we estimate

$$\begin{aligned}\left|F_{\sigma,\delta}(x) - f'(u_\delta)(x)\right| \;&\leq\; \int_{-R-2\delta}^{R_1}\left(\int_{-R-2\delta}^{y}\left|\tilde{h}_{\sigma,\delta}(z) - h_\delta(z)\right|dz\right)dy \\ &\leq\; \frac{(R_1+R+2\delta)^2}{2}\cdot\sigma \qquad \text{for all } x\in(-\infty,R_1].\end{aligned}$$

This also implies that

$$|\beta_1| \;=\; \left|F_{\sigma,\delta}(R_1) - f'(0)\right| \;=\; \left|F_{\sigma,\delta}(R_1) - f'(u_\delta)(R_1)\right| \;\leq\; \frac{(R_1+R+2\delta)^2}{2}\cdot\sigma.$$

Since $\left|G_\theta(x)\right| \leq \theta\cdot\left(4|\alpha_2| + \frac{|\alpha_3|}{2}\right)$ for all $x\in[R_1,R_1+\theta]$, one gets from (4.36) that

$$\left|F_{\sigma,\delta}(x) - f'(u_\delta)(x)\right| \;=\; \left|F_{\sigma,\delta}(x) - f'(0)\right| \;\leq\; \frac{(R_1+R+2\delta)^2}{2}\cdot\sigma + \theta\cdot\left(4|\alpha_2| + \frac{|\alpha_3|}{2}\right)$$

for all $x\in[R_1,\infty)$. Thus, we can choose $\theta>0$ sufficiently small such that

$$\|F_{\sigma,\delta} - f'(u_\delta)\|_{\mathbf{L}^\infty(\mathbb{R})} \;\leq\; 2(R+2\delta)^2\cdot\sigma.$$

3. Let $v_{\sigma,\delta}:\mathbb{R}\to\mathbb{R}$ be a $\mathcal{C}^2$ function such that

$$v_{\sigma,\delta}(x) \;=\; \left(f'\right)^{-1}\left(F_\delta(x)\right) \qquad \text{for all } x\in\mathbb{R}.$$

By the uniform convexity of f in (1.4), we get

$$|v_{\sigma,\delta}(x) - u_\delta(x)| \;\leq\; \frac{1}{\lambda}\cdot\left|F_{\sigma,\delta}(x) - f'(u_\delta)(x)\right| \;\leq\; \frac{2(R+2\delta)^2\cdot\sigma}{\lambda},$$

and (4.32) yields

$$\|v_{\sigma,\delta} - \bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; \|u_\delta - \bar{u}\|_{\mathbf{L}^1(\mathbb{R})} + \|v_{\sigma,\delta} - u_\delta\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; V\delta + \frac{4(R+2\delta)^3\cdot\sigma}{\lambda}.$$

Given $\varepsilon>0$, if we choose

$$\delta \;=\; \frac{\varepsilon}{2V} \qquad \text{and} \qquad \sigma \;=\; \frac{\lambda\varepsilon}{2^3(R+2\delta)^3},$$

then the function $\bar{v} := v_{\sigma,\delta}$ has $\mathrm{Supp}(\bar{v})\subseteq[-2R,2R]$ and $\|\bar{v}-\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\leq\varepsilon$. In the case, recalling (4.33), we have

$$\Psi_{h_\delta}(\sigma) \;=\; \min\left\{\frac{\lambda\varepsilon^4}{2^6 45 V^4(V+1)(R+2\delta)^3\left\|f\right\|_{C^3\left(-\frac{V}{2},\frac{V}{2}\right)}},\ \frac{2\varepsilon}{5V}\cdot\Psi^V_{f^{(3)}_{(-\frac{V}{2},\frac{V}{2})}}\left(\frac{5\lambda\varepsilon^3}{2^9V^4(R+2\delta)^3}\right)\right\}.$$

Thus, (4.37) yields

$$\begin{aligned}&\#\left\{x\in\mathbb{R} : x \text{ is an inflection point of } f'[\bar{v}]\right\}\\ &\quad\leq\; \max\left\{\frac{2^8 45 V^4(V+1)(R+2\delta)^4\left\|f\right\|_{C^3\left(-\frac{V}{2},\frac{V}{2}\right)}}{\lambda\varepsilon^4},\ \frac{10V(R+2\delta)}{\varepsilon\cdot\Psi^V_{f^{(3)}_{(-\frac{V}{2},\frac{V}{2})}}\left(\frac{5\lambda\varepsilon^3}{2^9V^4(R+2\delta)^3}\right)}\right\} + 4\\ &\qquad\qquad =\; 2^8\cdot\frac{V^5(R+2\delta)^4}{\lambda\varepsilon^4}\cdot\max\left\{45\left(1+\frac{1}{V}\right)\left\|f\right\|_{C^3\left(-\frac{V}{2},\frac{V}{2}\right)},\ \frac{4\beta_\varepsilon}{\Psi^V_{f^{(3)}_{(-\frac{V}{2},\frac{V}{2})}}(\beta_\varepsilon)}\right\} + 4\end{aligned}$$

with $\beta_\varepsilon = \dfrac{5\lambda\varepsilon^3}{2^9V^4R^3}$. In particular, for $0 < \varepsilon \leq \dfrac{RV}{4}$ such that $2\delta \leq R$, it holds

$$\#\big\{x \in \mathbb{R} : x \text{ is an inflection point of } f'[\bar{v}]\big\} \;\leq\; \frac{\Phi_{f,V,R}(\varepsilon)}{\lambda} \cdot \frac{R^4V^5}{\varepsilon^4} + 4$$

with $\Phi_{f,V,R}(\varepsilon)$ defined in (4.30).

4. To complete the proof, recalling Lemma 4.2, we obtain that the total number of shock curves in the weak entropy solution v of (1.3) with initial data $u(0,\cdot) = \bar{v}$ is bounded as in (4.31). □

Remark 4.4 *If we assume $f \in \mathcal{C}^4(\mathbb{R})$ as in Theorem 1.2, then from (2.9) it holds that*

$$\Psi^V_{f^{(3)}}(s) \;\geq\; \frac{s}{\|f\|_{C^4(-\frac{V}{2},\frac{V}{2})}} \qquad \textit{for all } s > 0.$$

Thus, the function $\Phi_{f,V,R}$ is bounded by

$$\Phi_{f,V,R} \;\leq\; C \;:=\; 2^{12}45 \cdot \left(1 + \frac{1}{V}\right) \cdot \|f\|_{C^4\left(-\frac{V}{2},\frac{V}{2}\right)} \tag{4.38}$$

and (4.31) yields (1.6).

Finally, in the spirit of approximation theory, we state the following corollary of Theorem 1.2.

Corollary 4.5 *Under the same assumptions in Theorem 1.2, given an integer $N > 4$ and an initial datum $\bar{u} \in \mathbf{L}^1(\mathbb{R})$ satisfied (1.5), there exists $\bar{v} \in \mathcal{C}^3(\mathbb{R})$ with $\mathrm{supp}(v) \subseteq (-2R, 2R)$ and*

$$\|\bar{v} - \bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; 2^3(45)^{1/4} \cdot \left[\frac{V+1}{\lambda} \cdot \|f\|_{C^4\left(-\frac{V}{2},\frac{V}{2}\right)}\right]^{1/4} \cdot \frac{RV}{(N-4)^{1/4}}$$

such that the entropy weak solution $v = v(t,x)$ of (1.3) with initial datum $v(0,\cdot) = \bar{v}$ contains at most N shocks.

Acknowledgments. This research by K. T. Nguyen was partially supported by a grant from the Simons Foundation/SFARI (521811, NTK). The authors would like to warmly thank Prof. Bressan for suggesting the problem during his talk at Penn State University.

References

[1] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Science Publications, Clarendon Press, Oxford, UK, (2000).

[2] F. Ancona, O. Glass, and K. T. Nguyen, Lower compactness estimates for scalar balance laws, *Comm. Pure Appl. Math. 65* (2012), no. 9, 1303–1329.

[3] F. Ancona, O. Glass, and K. T. Nguyen, On compactness estimates for hyperbolic systems of conservation laws, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32 (2015), no. 6, 1229–1257.

[4] F. Ancona, O. Glass, and K. T. Nguyen, On Kolmogorov entropy compactness estimates for scalar conservation laws without uniform convexity, *SIAM J. Math. Anal.* **51** (2019), no. 4, 3020–3051.

[5] J.M. Bloom, The local structure of smooth maps of manifolds. B.A. thesis, Harvard 2004. (Link: https://www.math.harvard.edu/media/ThesisXFinal.pdf)

[6] René Thom, Quelques propriétés globales des variétés differentiables, *Commentarii Mathematici Helvetici* **28** (1954), no. 1, 17-86.

[7] P. Bénilan P and M. G. Crandall. Regularizing effects of homogeneous evolution equations, *In Contributions to analysis and geometry (Baltimore, Md., 1980)*, pages 23–39. Johns Hopkins Univ. Press, Baltimore, Md., 1981.

[8] A. Bressan, *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy Problem.* Oxford University Press, Oxford 2000.

[9] R. Capuani, P. Dutta Prerona, and K. T. Nguyen, Metric entropy for functions of bounded total generalized variation, *SIAM J. Math. Anal.* **53** (2021), no. 1, 1168–1190.

[10] C. Dafermos, Generalized characteristics and the structure of solutions of Hyperbolic conservation laws, *Indiana Univ. Math. J.*, **26 (1977)**, no. 6, 1097-1119

[11] C. Dafermos and X. Geng, Generalized characteristics uniqueness and regularity of solutions in a hyperbolic system of conservation laws, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8** (1991), no. 3-4, 231–269.

[12] J. Damon, Generic properties of solutions to partial differential equations, *Arch. Ration. Mech. Anal.* **140** (1997), no. 4, 353–403.

[13] D. Schaeffer, A regularity theorem for conservation laws, *Adv. in Math.* **11** (1973), 368–386.

[14] R. J. DiPerna, Singularities of solutions of nonlinear hyperbolic systems of conservation laws, *Arch. Rational Mech. Anal* **60** (1975), 75-100.

[15] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure. Appl. Math* **18** (1965), 697-715.

[16] J. Glimm, Decay of solutions of nonlinear hyperbolic conservation laws, *Mem. Amer. Math. Soc* **101** (1970).

[17] A.N. Kolmogorov and V.M Tikhomirov, ε-entropy and ε-capacity of sets in functional spaces, *Uspekhi Mat. Nauk* **14** (1959), 3–86.

[18] O.A. Oleinik, Discontinuous solutions of nonlinear-differential equation, *Usp. Math. Nauk (N.S.)* **12** (3) (1957), 3-73. English translation: Amer. Math. Soc. Transl., Ser. 2, **26**, 95-172.

[19] O.A. Oleinik, The Cauchy problem for nonlinear equations in a class of discontinuous functions, *Dokl. Akad. Nauk SSSR* **95** (1954), 451-454. English translation: Amer. Math. Soc. Transl. Ser. 2, **42**, 7-12.

[20] O.A. Oleinik, The Cauchy problem for nonlinear differential equations of the first order with discontinuous initial conditions, *Trudy Moskov. Mat. Obsc.* **5** (1956), 433-454.